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ARITHMETICAL DEDUCTION OF KRONECKER'S CLASS-NUMBER RELATIONS.

By G. H. Cresse.

The class-number relations which appear near the end of this paper are three of the eight celebrated class-number recursion formulas which L. Kronecker published[*] in 1860. In a preliminary announcement,[†] he had said, "If n denote an odd number > 3 and κ denote the modulus of an elliptic function, then the number of different values of κ^2 for which multiplication of the elliptic function by $\sqrt{-n}$ is possible is six times the number of classes of quadratic forms belonging to the determinant $-n$. Each value of κ^2 is the root of an integral equation whose degree is the number of such values of κ^2." Later[‡] he intimated clearly that his only method of obtaining class-number relations from the theory of singular moduli was by setting two moduli equal to each other in the modular equation. By this method, H. J. Stephen Smith deduced[§] in detail the eight formulas in a report which Kronecker has commended[||] for insight and mastery of principles.

C. Hermite[¶] showed how a class-number relation can be obtained by equating the coefficients of like powers of $e^{\pi i\omega}$ in two expansions of a "doubly periodic function of the third kind." K. Petr[**] by Hermite's method deduced all eight of Kronecker's relations. In parallel researches, G. Humbert[††] and L. J. Mordell[‡‡] have reproduced independently many of Petr's intermediate results.

Kronecker[§§] set up a one-to-one correspondence between certain quadratic forms and bilinear forms in four variables and then developed a theory of bilinear forms which established arithmetically the first six of his eight formulas. More interest however has been taken in the method of arithmetical deduction which was first illustrated by J. Liouville[*] and which has been applied to a certain class-number relation by L. J. Mordell.[†] The method is the result of translating one of Hermite's analytic proofs into an arithmetical one. J. V. Uspensky[‡] has accomplished the deduction of the eight formulas by this method. In the remainder of this paper, I shall reproduce the substance of his proof of formulas I, II, V and supply myself many desirable details including the proofs of his lemmas.

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[*] Jour. für Math., 57, 1860, 248–255; Jour. de math. (2), 5, 1860, 289–299.

[†] Monatsberichte Akad., Berlin, Oct., 1857, 456.

[‡] Ibid., 1875, 235.

[§] Report of the British Association, 35, 1865, 349–359.

[||] Monatsberichte Akad., Berlin, 1875, 234.

[¶] Comptes Rendus, Paris, 53, 1861, 214–228; Jour. de math. (2), 7, 1862, 25–44; Œuvres, Paris, 1908, II, 109–124.

[**] Rozpravy ceske Akademia, Prague, 9, 1900, No. 38 (Bohemian language).

[††] Jour. de math. (6), 3, 1907, 337–449.

[‡‡] Messenger of Math., 45, 1916, 76–80.

[§§] Monatsberichte Akad., Berlin, 1866, 873; Abhandlungen Königl. Preuss. Akad. Wiss., Berlin, 1883, II, 2d Abhand., pp. 60. Werke, Leipzig, 1897, II, 425–490.

LEMMA 1. Let $F(x)$ be an uneven arithmetical function of the integer x; i.e., $F(-x) = -F(x)$, $F(0) = 0$. Let m be an uneven positive number and let

$$\sum F\left(\frac{d+\delta}{2}\right) = S, \tag{1}$$

where the summation extends to all integer solutions of the equation

$$m = 4h^2 + d\delta, \tag{2}$$

in which d and δ are positive but $h \gtreqless 0$. Let

$$\sum F(i + d') = S', \tag{3}$$

where the summation extends to all integer solutions of the equation

$$m = i^2 + 2d'\delta', \tag{4}$$

in which d' and δ' are positive and δ' uneven, but i is $\gtrless 0$. Then $S = 2S'$ if m is not a square, and $S = 2S' + \sqrt{m}\, F(\sqrt{m})$ if m is a square. That is,

$$\sum_{m=4h^2+d\delta} F\left(\frac{d+\delta}{2}\right) = 2 \sum_{\substack{m=i^2+2\delta'd' \\ \delta' \equiv 1 \pmod 2}} F(i+d') + \begin{cases} 0, \text{ if } m \text{ is not a square,} \\ \sqrt{m}F(\sqrt{m}), \text{ if } m \text{ is a square.} \end{cases} \tag{A}$$

Proof[§] *of Lemma* 1. Referring to (3) and (4), let

$$x = i + d', \qquad y = \delta' - i, \qquad z = i + d' - \delta';$$

that is

$$i = x - y - z, \qquad d' = y + z, \qquad \delta' = x - z.$$

Then (4) becomes all representations

$$m = x^2 + y^2 - z^2, \tag{5}$$

in which x and z are each even or uneven $\gtreqless 0$; y even $\gtreqless 0$; $y + z > 0$; $x > z$. But the sum (3) is not affected if we add the condition $x + z > 0$; for, corresponding to every solution (x, y, z) of (5) in which $x + z$ is < 0, there is a solution $(-x, y, z)$ in which $x + z$ is < 0. Hence we consider in (5) all and only the representations in which

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[*] Jour. de math. (2), 7, 1862, 44–48. Details of proof have been furnished by H. J. S. Smith, Report British Assoc., 35, 1865, 366–369; and by P. Bachmann, Niedere Zahlentheorie, Leipzig, 1910, II, 423–433.

[†] Messenger of Mathematics, 45, 1916, 177–180.

[‡] Math. Sbornik, Moscow, 29, 1913, 26–52 (Russian language).

[§] Cf. H. J. Stephen Smith, Rep. Brit. Assoc., 35, 1865, 368.

$$y + z > 0, \qquad x > |z|. \tag{6}$$

That is to say, in (3), S' is equal to $\sum F(x)$ in which the summation extends to all solutions (x, y, z) of (5), (6).

Referring to (1) and (2), let

$$x = \frac{1}{2}(d + \delta), \qquad z = \frac{1}{2}(d - \delta), \qquad y = 2h.$$

Then (2) becomes all the representations

$$m = x^2 + y^2 - z^2, \tag{7}$$

in which y is even and $\gtreqless 0$ and

$$x > |z|. \tag{8}$$

That is to say, in (1), S is equal to $\sum F(x)$ in which the summation extends to all solutions (x, y, z) of (7), (8).

(*a*) *Case of m a non-square,* i.e., $|y| \neq |z|$. If $(x > 0, y > 0, z < y)$ is a solution of (5), (6), then $(x, y, -z)$ but neither $(x, -y, z)$ nor $(x, -y, -z)$ is a solution of (5), (6); while all four sets are solutions of (5), (6), while all four sets are solutions of (7), (8).

If $(x > 0, y > 0, z > y)$ is a solution of (5), (6), then $(x, -y, z)$ but neither $(x, y, -z)$ nor $(x, -y, -z)$ is a solution of (5), (6), while all four sets are solutions of (7), (8).

If $(x > 0, y < 0, z > y)$ is a solution of (5), (6), then $(x, -y, z)$ but neither $(x, y, -z)$ nor $(x, -y, -z)$ is a solution of (7), (8).

The categories given in the last three paragraphs of solutions of (7), (8) are exhaustive. Hence the lemma is proved for m a non-square.

(*b*) *Case of m a square.* To the solutions of (5), (6) for case (*a*), there will now be added only those solutions (x, y, z) in which $y = z > 0$. But corresponding to each of these new solutions of (5), (6) there are the new solutions (x, y, z), $(x, -y, z)$, $(x, y, -z)$, $(x, -y, -z)$ of (7), (8). The number of such solutions $(x, y, -z)$ and $(x, -y, z)$ combined with the solution $(x, 0, 0)$ of (7), (8) is $\sqrt{m}$ and the sum of the corresponding terms of (1) is $\sqrt{m}\, F(\sqrt{m})$. This completes the proof of *Lemma* 1.

Let $f(x)$ be an even function of the integer x. Let σ be an arbitrary real number. Then the function

$$F(x) = f(x - \sigma) - f(x + \sigma)$$

is an uneven function of (x). In (A), we replace m by $m - 2\rho\sigma$, where ρ and σ are given uneven positive numbers; and we take F as just defined. Then (A) becomes

$$\sum_{\substack{d,\,\delta>0\\ m=2\rho\sigma+4h^2+d\delta\\ h\gtreqless 0}} \left[f\left(\frac{d+\delta}{2}-\sigma\right)-f\left(\frac{d+\delta}{2}+\sigma\right)\right]$$

$$= 2 \sum_{\substack{i,\,d\\ i\gtreqless 0,\, d>0,\\ m=2\rho\sigma+i^2+2d\delta\\ \delta\equiv 1 \pmod 2),\, >0}} [f(i+d-\sigma)-f(i+d+\sigma)] \tag{B}$$

$$+\begin{Bmatrix} 0 \text{ if } m-2\rho\sigma \text{ is not a square} \\ s[f(s-\sigma)-f(s+\sigma)] \text{ if } m-2\rho\sigma=s^2>0 \end{Bmatrix}.$$

We take hereafter in this paper $m = 4n+1$ and for this case the brace in the last equation is equal to zero, since $m-2\rho\sigma = s^2$ has no solution for odd ρ and σ. We may now extend the summation in (*B*) to all possible ρ and σ, and have:

$$\sum_{\substack{d,\,\delta,\,\sigma>0\\ m=4h^2+2\rho\sigma+d\delta\\ h\gtreqless 0,\,\rho>0\\ \rho\equiv\sigma\equiv 1 \pmod 2)}} \left[f\left(\frac{d+\delta}{2}-\sigma\right)+f\left(\frac{d+\delta}{2}+\sigma\right)\right]$$

$$= 2\cdot \sum_{\substack{i,\,d,\,\sigma>0\\ m=2\rho\sigma+i^2+2d\delta\\ \delta,\,\rho>0,\,\delta\equiv\rho\equiv\sigma\equiv 1 \pmod 2)}} [f(d-\sigma+i) \tag{C}$$

$$+f(d-\sigma-i)-f(d+\sigma+i)-f(d+\sigma-i)].$$

The right member of (*C*) is transformed by means* of

LEMMA 2. Let n be an even number and consider all the representations

$$n = d\delta+\rho\sigma,$$

in which d, δ, ρ, σ are positive uneven numbers; also consider all the representations

$$n = d\delta,$$

in which d, δ are positive integers, δ uneven. If $\psi(x)$ is an even function, then

$$2\sum_{\substack{\rho,\,\sigma,\,d,\,\delta>0,\\ \equiv 1 \pmod 2)\\ n=\rho\sigma+d\delta}} [\psi(d-\sigma)-\psi(d+\sigma)] = \sum_{\substack{d>0\\ n=d\delta\\ \delta>0,\,\equiv 1 \pmod 2)}} d[\psi(0)-\psi(d)].$$

Proof† *of Lemma* 2. Consider the system

$$d\delta'+\rho'\sigma = n, \qquad d+\sigma = 2\mu, \qquad \delta'-\rho' = 2\nu; \tag{a}$$

in which d, δ' ρ', σ are positive and uneven, and μ, ν are given positive integers. The solutions of this system are equal in number to the solutions of the system

$$d\delta'+\rho'\sigma = n, \qquad d-\sigma = -2\mu, \qquad \delta'+\rho' = 2\nu. \tag{a'}$$

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* Cf. J. Liouville, Jour. de math. (2), 3, 1858, 194.

† Cf. H. J. S. Smith, Rep. Brit. Assoc., 35, 1865, 366–367.

For, eliminating δ' and σ, we find that (a) has as many solutions as $n/2 = \nu d + \mu\rho'$ has solutions in which $d < 2\mu$ and (a') has as many as the same equation has solutions in which $\rho' < 2\nu$. These numbers of solutions are the same. For, corresponding to each solution of $n/2 = \nu d + \mu\rho'$ in which $d < 2\mu$ and $\rho' > 2\nu$, there is a solution of $n/2 = \nu(d + 2k\mu) + \mu(\rho' - 2k\nu)$ in which $\rho' - 2k\nu < 2\nu$; k being so chosen that $2k\nu$ is the largest multiple of 2ν that is $< \rho'$.

Similarly, the solutions of the two systems

$$d\delta' + \rho'\sigma = n, \qquad d + \sigma = 2\mu, \qquad \delta' - \rho' = -2\nu; \tag{b}$$

and

$$d\delta' + \sigma\rho' = n, \qquad d - \sigma = -2\mu, \qquad \delta' + \rho' = 2\nu, \tag{b'}$$

are equal in number. Also the number of solutions of each of the two systems

$$d\delta' + \rho'\sigma = n, \qquad d + \sigma = d'', \qquad \delta' = \rho' = \delta''; \tag{c}$$

and

$$d\delta' + \rho'\sigma = n, \qquad d = \sigma = \delta'', \qquad \delta' + \rho' = d'', \tag{c'}$$

for each pair of conjugate divisors d'', δ'' of n, is $\dfrac{d''}{2}\cdot$

Hence if $\psi(x)$ is an even function, the enumeration of the solutions of (a), (a'), (b), (b'), (c), (c') gives the *Lemma* 2.

We specialize the even function $\psi(x)$ as

$$\psi(x) = f(x - i) + f(x + i)$$

in which $f(x)$ and i have the same meanings respectively as above. So the *Lemma* 2 applied to the right member of (C) becomes

$$2 \sum_{\substack{i,\, d,\, \sigma > a \\ d \equiv \delta \equiv \rho \equiv \sigma \equiv 1 \,(\mathrm{mod}\, 2) \\ \frac{m - i^2}{2} = \rho\sigma + d\delta}} [f(d - \sigma + i) + f(d - \sigma - i)$$

$$- f(d + \sigma + i) - f(d + \sigma - i)]$$

$$= \sum_{\substack{d,\, \delta,\, i > 0 \\ \delta \equiv 1 \,(\mathrm{mod}\, 2) \\ \frac{m - i^2}{2} = d\delta}} d[2f(i) - f(d + i) - f(d - i)].$$

And hence (C) becomes

$$\sum_{\substack{d,\, \delta,\, \rho,\, \sigma > 0 \\ \rho \equiv \sigma \equiv 1 \,(\mathrm{mod}\, 2) \\ m = 4h^2 + 2\rho\sigma + d\delta \\ h \gtreqless 0}} \left[f\left(\frac{d + \delta}{2} - \sigma\right) - f\left(\frac{d + \delta}{2} + \sigma\right)\right]$$

$$= \sum_{\substack{i,\, d > 0 \\ m = i^2 + 2d\delta \\ \delta > 0 \\ \equiv 1 \,(\mathrm{mod}\, 2)}} d[2f(i) - f(d + i) - f(d - i)]. \tag{C'}$$

From (C'), Kronecker's classical formulas I, II, V now follow by taking $f(\pm 1) = 1$, $f(x) = 0$ if x^2 is not $= 1$.

We evaluate the left member of (C'). Only the first f has significance and that only for the argument $\frac{d+\delta}{2} - \sigma = \pm 1$. We denote the uneven number $\frac{d-\delta}{2}$ by τ and take first $\frac{d+\delta}{2} - \sigma = +1$. It follows that $-\sigma \leqq \tau \leqq \sigma$. Consequently, if we set

$$2u = \sigma - \tau, \qquad 2v = \sigma + \tau,$$

the numbers u and v will be $\geqq 0$ and of different parity. Then since $m = 4n + 1$, it is easily found, when we set $\rho = 2k - 1$, that the above equation, $m = 4h^2 + 2\rho\sigma + d\delta$, is equivalent to either of the following three equations: *

$$\begin{aligned} n - h^2 &= (k + u)(k + v) - k^2, \\ n - h^2 &= (u + k)(u + v) - u^2, \\ n - h^2 &= (u + v)(u + k) - u^2, \end{aligned} \qquad (D)$$

$h \gtreqless 0$, $k \geqq 1$, $u \geqq 0$, $v \geqq 0$, $u + v \equiv 1 \pmod 2$.

It is evident that the complete number of solutions of each equation in (D) is 2 times the number of those solutions in which u is $< v$. Hence we confine our study to these latter solutions of (D). To each solution of (D_1) in which $k \leqq u < v$, there corresponds a quadratic form (A, B, C) in which

$$A = u + k, \qquad B = k, \qquad C = v + k,$$

of determinant $h^2 - n < 0$, whose coefficients satisfy the condition

$$A < C, \qquad B > 0, \qquad 2B \leqq A, \qquad A + C \equiv 1 \pmod 2.$$

To each of the solutions of (D_2) in which $u < k \leqq v$, there corresponds a quadratic form (A, B, C) in which

$$A = u + k, \qquad B = u, \qquad C = u + v,$$

of determinant $h^2 - n < 0$, whose coefficients satisfy the condition

$$A \leqq C, \qquad B \geqq 0, \qquad 2B < A, \qquad C \equiv 1 \pmod 2.$$

To each of the solutions of (D_3) in which $u < v < k$, there corresponds a quadratic form (A, B, C) in which

$$A = u + v, \qquad B = u, \qquad C = u + k,$$

of determinant $h^2 - n < 0$, whose coefficients satisfy the condition

$$A < C, \qquad B \geqq 0, \qquad 2B < A, \qquad A \equiv 1 \pmod 2.$$

Conversely, to an arbitrary form (A, B, C) of any of the three types

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* Cf. C. Hermite, Jour. de math. (2), 7, 1862, 32; Œuvres, Paris, 2, 1908, 116.

just considered, there corresponds uniquely a solution u, v, k, of (D). Hence the number of solutions of the above equation $m = 4h^2 + 2\rho\sigma + d\delta$ is

$$4P + 2Q + 2R + 2S,$$

in which P, Q, R, S denote numbers of forms (A, B, C) of determinant $h^2 - n < 0$:

P, the number of those forms in which $A < C$, $B > 0$, $2B < A$ and one of the numbers A and C is uneven;

Q, the number of those forms in which $A < C$, $2B = A$ and one of the numbers A and C is uneven;

R, the number of those forms in which $A < C$, $B = 0$ and one of the numbers A and C is uneven;

S, the number of those forms in which $A = C$, $B \geqq 0$, $2B < A$, $A \equiv 1 \pmod 2$.

Similarly, if we take $\frac{d + \delta}{2} - \sigma = -1$ in (C), it is found that the number of solutions of $m = 4h^2 + 2\rho\sigma + d\delta$ is

$$4P + 2Q + 2T + 2R;$$

in which P, Q, R have the same meaning as before, and T denotes the number of forms (A, B, C) of determinant $h^2 - n < 0$, satisfying the conditions

$$A = C, \qquad B > 0, \qquad 2B < A, \qquad A \equiv 1 \pmod 2.$$

But $8P + 4Q + 4R + 4S$ is the quadruple of the number of uneven classes of determinant $h^2 - n$. Denoting by $F(\Delta)$ the number of such classes of determinant $-\Delta$, we find then that the left member of (C') has the value

$$4 \sum_{\leqq \sqrt{n}} F(n - h^2) + 2T - 2S.$$

Now $2T - 2S = 0$, except when $n - h^2$ is the square of an uneven number and for such a value of $n - h^2$, $2T - 2S = -2$. Hence the left member of (C'), by our choice of the function $f(x)$, has the value

$$4\sum_h F(n - h^2) - 2\sigma(n),$$

in which the summation extends to all integral values of $h (\gtreqless 0)$ whose squares are $\leqq n$, and $\sigma(n)$ denotes the number of all representations of n in the form

$$n = s^2 + h^2,$$

where s is uneven and positive.

We evaluate the right member of (C'), namely:

$$2\sum df(i) - \sum df(d - i) - \sum df(d + i),$$

$$m = i^2 + 2d\delta, \qquad d, i > 0, \qquad \delta \equiv 1 \pmod 2), > 0.$$

In view of our choice of the function $f(x)$, the significant terms in the first sum are all and only those in which $i = 1$; those in the second sum have $i = d \pm 1$; the terms in the third sum are all zero.

The significant terms of the first sum correspond respectively to the solutions of

$$m = 4n + 1 = 1 + 2d\delta,$$

that is, to the solutions of

$$n = d'\delta, \qquad d = 2d',$$

and therefore the first sum is

$$2^{\alpha+2}X(n), \tag{E}$$

where $X(n)$ denotes the sum of the uneven divisors of n and 2^α is the highest power of 2 contained in n.

The terms of the second sum correspond respectively to solutions of

$$m = 4n + 1 = (d \pm 1)^2 + 2d\delta = d(d \pm 2 + 2\delta) + 1;$$

that is, to the solutions of

$$n = d'(d' \pm 1 + \delta), \qquad d = 2d'.$$

Hence the second sum will be represented by

$$2 \sum_{n = \Delta\Delta';\ \Delta' \leqq \Delta} \Delta' + 2 \sum_{n = \Delta\Delta';\ \Delta' < \Delta} \Delta', \tag{F}$$

where the summations are extended to positive integers Δ and Δ' of the same parity.

(*a*) Suppose that n is of the form $4r$. Then the sum in (*E*) has the value

$$2^3[X(r) + (2^\alpha - 1)X(r)] = 2^3[X(r) + (1 + 2 + 2^2 + \cdots 2^{\alpha-2})X(r)]$$
$$= 2^3[X(r) + \Phi(r)],$$

where $\Phi(r)$ denotes the sum of the divisors of r.

The total sum expressed in (*F*) is now

$$8\Theta(r) + 4\epsilon\sqrt{r},$$

where $\Theta(r)$ denotes the sum of the divisors of r which are $< \sqrt{r}$ and $\epsilon = 1$ or 0 according as r is or not a square. We write

$$4\Theta(r) = 4Z(r) - 4\Psi(r),$$

where $Z(r)$ denotes the sum of the divisors of r which are $> \sqrt{r}$ and $\Psi(r)$ is defined by the identity. Moreover, by definition

$$4\Theta(r) + 4e\sqrt{r} = 4\Phi(r) - 4Z(r).$$

Combining the last two identities, we have for (*F*) the expression

$$4[\Phi(r) - \Psi(r)].$$

Since $\sigma(4r) = 0$, (C') now implies Kronecker's first class-number relation

$$F(4r) + 2F(4r - 1^2) + 2F(4r - 2^2) + \cdots = 2X(r) + \Phi(r) + \Psi(r).$$

(*b*) Suppose that n is of the form $2s$, s uneven. The sum in (E) will be $8\Phi(s)$; the sum in (F) will be lacking. The arithmetical function $\sigma(2s)$ is double the excess* of the number of divisors of s which have the form $4k + 1$ over that of divisors which have the form $4k - 1$. When we denote this difference $\left(= \sum_{\delta/r}(-1)^{\frac{\delta-1}{2}}\right)$ by $\varphi(s)$, (C') gives Kronecker's second class-number relation

$$F(2s) + 2F(2s - 1^2) + 2F(2s - 2^2) + \cdots = 2\Phi(s) + \varphi(s).$$

(*c*) Suppose that n is the uneven number s. The right member of (C') is now $2\Phi(s) + 2\Psi(s)$; and $\sigma(s) = \varphi(s)$. Hence (C') implies Kronecker's fifth class-number relation

$$F(s) + 2F(s - 1^2) + 2F(s - 2^2) + \cdots = \tfrac{1}{2}[\Phi(s) + \Psi(s) + \varphi(s)].$$

In a similar deduction of Kronecker's formulas III, IV and VI, the analog of the above *Lemma* 1 is the following for $m = 4n + 1$:

$$\sum_{m = 4h^2 + d\delta} (-1)^h F\left(\frac{d + \delta}{2}\right) = 2 \sum_{\substack{m = i^2 + 2d'\delta' \\ \delta' \equiv 1 \,(\mathrm{mod}\ 2)}} (-1)^{\frac{i-1}{2} + \frac{\delta-1}{2}} F(i+d');$$

in which the denotations are the same as in *Lemma* 1. The analog of *Lemma* 2 is here

$$\sum_{\substack{\tau = \rho\sigma + d\delta \\ \rho,\, \sigma,\, d,\, \delta \equiv 1 \,(mod\ 2)}} (-1)^{\frac{\delta-1}{2}}[\Theta(d + \sigma) - \Theta(d - \sigma)] = \sum_{\substack{\tau = d\delta \\ \delta \equiv 1 \,(\mathrm{mod}\ 2)}} (-1)^{\frac{\delta-1}{2}} d\Theta(2d),$$

in which the denotations are the same as in *Lemma* 2, except that $\Theta(x)$ is an arbitrary uneven function; $\Theta(0) = 0$; and τ is even.

By setting $\Theta(x) = f(x + i) - f(x - i)$, where $f(x)$ is an arbitrary even function of x, the following analog of (C') is obtained:

$$\sum_{\substack{\sigma,\, d,\, \delta > 0 \\ \equiv 1 \,(\mathrm{mod}\ 2) \\ m = 4h^2 + 2\rho\sigma + d\delta \\ h \gtreqless 0 \\ \rho \equiv 1 \,(\mathrm{mod}\ 2) \\ > 0}} (-1)^h \left[f\left(\frac{d + \delta}{2} - \sigma\right) - f\left(\frac{d + \delta}{2} + \sigma\right)\right]$$

$$= 2 \cdot \sum_{\substack{i,\, d,\, \delta > 0 \\ \delta \equiv 1 \,(\mathrm{mod}\ 2) \\ m = i^2 + 2d\delta}} (-1)^{\frac{i-1}{2} + \frac{\delta-1}{2}} d[f(2d - i) - f(2d + i)].$$

Two other similar pairs of lemmas lead respectively to Kronecker's formulas VII, VIII.

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[*] Cf. L. E. Dickson, History of the Theory of Numbers, vol. II, p. 235.